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On an Extension of The Galerkin Method To Nonconservative Stability Problems

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ON AN EXTENSION OF
THE GALERKIN METHOD TO
NONCONSERVATIVE STABILITY PROBLEMS

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INTRODUCTION

Galerkin's method has proved to be a powerful technique for constructing approximate solutions of boundary value problems of engineering science. Following Mikhlin [1], a solution is assumed in the form of a known set of functions with unknown generalized co-ordinates or constants. The series, when substituted in the differential equation, gives rise to an error or residue which is required to be orthogonal to the members of the set. This results in a set of equations for determining the unknowns. The development, in recent years, of several closely related methods, such as the least square technique, the collocation method, and the method of moments, indicates the success and a general sense of confidence in the application of the concept of Galerkin.

This concept was originally introduced for solving self-adjoint boundary value problems for which a general proof of convergence has been supplied by Mikhlin [1]. Bolotin [2] has, however, suggested that by taking the eigenfunctions of free vibration as the coordinate functions, nonself-adjoint problems of elastic stability with follower-type loads may also be approximately solved by the Galerkin method. In this way he has successfully treated a number of nonconservative stability problems. But it appears that more effective generalizations of the Galerkin method in this area of nonconservative elastic stability are possible, and, in particular, studies which dwell on the question of convergence, estimation of error, and direct efficient use of digital computers are of interest.

The present study discusses an extension of the Galerkin method for direct numerical treatment of a class of nonconservative stability problems. For this purpose we consider an additional system by combining the force field of the adjoint to the original problem. The authors have recently proposed variational methods in which the concept of adjoint systems was employed [3,4]. We prove first that for a certain class of systems the resulting boundary value problem is self-adjoint. Therefore, although both fields of forces are separately

nonconservative, the combination of these is a conservative system of forces. This feature was first noted by Nemat-Nasser and Herrmann [5] who combined the force fields of Beck's and Reut's models [2]. This self-adjoint system bears close relationships to the original system, such that in the Galerkin method the coordinate functions may be selected as being the eigenfunctions of this resulting system. By means of this set of eigenfunctions, it is possible to transform the boundary value problem into an integral equation which may be solved numerically by quadrature.

The essence of the proposed extension of the Galerkin method for the class of nonself-adjoint system selected for study may be described as follows In an extended sense, the Galerkin method requires the solution of

$$\langle \mathcal{L}u \, , \, v \rangle = \langle f \, , \, v \rangle$$

for the problem $\mathcal{L}u = f$ with appropriate boundary conditions, where the unknown functions u is to be approximated by the coordinate functions If we select v such that the above may be reduced to

$$\langle \mathcal{L}^* v \, , \, u \rangle = \langle f \, , \, v \rangle$$

then this integral equation may be solved by means of a quadrature whose convergence property is known and certain error estimations may also be given. The nature of $\mathcal{L}^*$, the properties of v and certain quadrature schemes are discussed below.

Please note the difference between: L (capital ℓ)
and
$\mathcal{L}$ (script ℓ)

A CLASS OF NONCONSERVATIVE STABILITY PROBLEMS

We consider the following form of an ordinary, linear, differential equation:

$$\mathcal{L}u + \beta \mathcal{L}_1 u = \lambda u \tag{1}$$

where

$$\mathcal{L} = \sum_{n=1}^{N} \alpha_{N-n}(x) \frac{d^n}{dx^n} \tag{2}$$

$$\mathcal{L}_1 = \sum_{n=1}^{K} \beta_{k-n} \frac{d^n}{dx^n} \quad , \qquad K \leq N \tag{3}$$

In the above u denotes a function of a real variable x for $a \leq x \leq b$, α_n are continuous functions of x whose $N-n$ derivatives with respect to x exist and are continuous, and β_n are certain constants. β is a parameter representing the magnitude of the follower load. Further, α_0 does not vanish at any point of the closed interval (a, b).

Associated with (1) we consider N linear, homogeneous, boundary conditions in $u(a)$, $u'(a)$, ..., $u^{(N-1)}(a)$, $u(b)$, ..., $u^{(N-1)}(b)$, as given by

$$L_j u \equiv \sum_{n=0}^{N-1} \eta_{jn} \frac{d^n u}{dx^n} = 0 \quad , \qquad j = 1, 2, \ldots, N \tag{4}$$

η_{jn} are quantities characterizing certain properties, such as stiffness or inertia at the end points (a, b). For future use let us define N additional forms $L_{N+1}u$, ..., $L_{2N}u$ in $u^i(a)$ and $u^i(b)$, $i = 0, 1, \ldots, N-1$, so that $L_1 u$, $L_2 u$, ..., $L_{2N}u$ are linearly independent.

In several nonconservative structural stability problems it turns out that the follower load influences the governing equations of motion by the presence of the operator L_1 in (1) so that in the absence of such loadings the free vibration of the system is expressed by the same operator

$$Lv = \omega v \tag{5}$$

with the same boundary conditions

$$L_j u = 0 \quad , \quad j = 1 , 2 , \ldots , N \tag{6}$$

The eigenvalue problem governed by (5-6) is known to be self-adjoint and thus, the presence of the operator L_1 in (1) destroys this property. It is well to emphasize that the class of systems selected for study is such that the follower load does not affect the boundary conditions. It is further noted that this class admits the operator L_1 with constant coefficients. In a later section we will investigate the problem by rela ing this restriction and it will be shown that several systems with L_1 having variable coefficients also possess the same property.

From (1) we obtain, after integrating by parts [6],

$$\begin{aligned} &\langle u^* , (L + \beta L_1)u\rangle - \langle u , (L^* + \beta L^*_1)u^*\rangle \\ &= [P(u , u^*)]_a^b + \beta[P (u , u^*)]_a^b \end{aligned} \tag{7}$$

where

$$\langle u , v\rangle = \int_a^b uv dx$$

denotes spatial average and

$$L^{*}u^{*} = (-1)^{N} \frac{d^{N}(\alpha_0 u^{*})}{dx^{N}} + (-1)^{N-1} \frac{d^{N-1}(\alpha_1 u^{*})}{dx^{N-1}} + \ldots - \frac{d(\alpha_{N-1} u^{*})}{dx} \qquad (8)$$

$$L_1^{*}u^{*} = (-1)^{K}\beta_0 \frac{d^{K}u^{*}}{dx^{K}} + (-1)^{K-1}\beta_1 \frac{d^{K-1}u^{*}}{dx^{K-1}} + \ldots -\beta_{K-1} \frac{du^{*}}{dx} , \; K \leq N \qquad (9)$$

and $[P(u , u^{*})]_a^b$ and $[P_1(u , u^{*})]_a^b$ are bilinear forms in $u(a)$, $u'(a)$, ... , $u^{(N-1)}(a)$, $u(b)$, ... , $u^{(N-1)}(b)$, $u^{*}(a)$, ... , $u^{*(N-1)}(b)$.

By insisting on the property of system (5-6) to be self-adjoint, it may be stated that P must be expressed in the following form:

$$[P(u , u^{*})]_a^b = \sum_{n=1}^{2N} L_n u L_{2N+L-n} u^{*} \qquad (10)$$

and further that

$$L^{*} = L \qquad (11)$$

An examination of (8) reveals that a necessary condition for equation (11) to be valid is that N must be an even integer. In other words, the order of the differential equation must be even.

It is now possible to properly select $2K$ linearly independent forms M_1 , M_2 , ... , M_{2K} in $u^{*}(a)$, $u^{*1}(a)$, ... , $u^{*(K-1)}(a)$, $u^{*}(b)$, ... , $u^{*(K-1)}(b)$ so that P_1 may be written in the following form

$$[P_1(u , u^{*})]_a^b = \sum_{n=1}^{2K} L_n u \, M_{2K+1-n} u^{*} , \; K \leq N \qquad (12)$$

Therefore, an adjoint to system (1-4) may be defined by the following eigenvalue problem

$$Lu^* + \beta L_1^* u^* = \lambda^* u^* \tag{13}$$

with the following boundary conditions

$$L_j u^* + \beta M_j u^* = 0 \quad , \quad j = 1 , 2 , \ldots , N \tag{14}$$

Note that $M_{K+1} = M_{K+2} = \ldots = M_N = 0$.

It has been shown in past studies [3,4], that the two sets of eigenvalues $\{\lambda_i\}$ and $\{\lambda_i^*\}$ are identical. Note that the system (1-4) will be self-adjoint only if $M_j u^* \equiv 0$ and $L_1^* = L_1$, which is not the case for nonconservative stability problems.

A SELF-ADJOINT SYSTEM

Keeping in mind the developments of the previous section, one may hope to recover another system which is self-adjoint and whose eigenvalues have close relationships with those of the original problem. L_1 represents the field of the follower load whose adjoint force field is represented by L_1^* . Generally, this concept of the adjoint force field in nondissipative, nonconservative systems is quite abstract and its physical significance may be traced only in simple problems. Indeed, such a relationship between Beck's and Reut's models was discovered with some surprise by Nemat-Nasser and Herrmenn [5] who also pointed out that a combination of the two nonconservative forces results in a conservative system. In this section we examine a similar property mathematically for the group of problems under consideration.

We define an eigenvalue problem by the following differential equation

$$Lv + \beta L_1 v + \beta L_1^* v = \omega v \tag{15}$$

with the boundary conditions

$$L_j v + \beta M_j v = 0 \qquad , \qquad j = 1 , 2 , ... , N \tag{16}$$

and prove that it is a self-adjoint system.

After integrating by parts, one obtains

$$\begin{aligned} <v^* , (L + \beta L_1 + \beta L_1^*)v> &= <v , (L + \beta L_1^* + \beta L_1)v^*> + [P(v , v^*)]_a^b \\ &+ \beta[P_1(v , v^*)]_a^b + \beta[P_2(v , v^*)]_a^b \end{aligned} \tag{17}$$

The bilinear concomitants P_1 and P_2 may be written explicitly as

$$\begin{aligned} P_1(v , v^*) = v[\beta_{K-1}v^* - \beta_{K-2}\frac{dv^*}{dx} + ... + (-1)^{K-1}\beta_0 \frac{d^{K-1}v^*}{dx^{K-1}}] \\ + \frac{dv}{dx}[\beta_{K-2}v^* - \beta_{K-3}\frac{dv^*}{dx} + ... + (-1)^{K-2}\beta_0 \frac{d^{K-2}v^*}{dx^{K-1}}] \\ + ... + \frac{d^{K-1}v}{dx^{K-1}}\beta_0 v^* \end{aligned} \tag{18}$$

and

$$\begin{aligned} P_2(v , v^*) = v[-\beta_{K-1}v^* - \beta_{K-2}\frac{dv^*}{dx} + ... + (-1)^{2K-1}\beta_0 \frac{d^{K-1}v^*}{dx^{K-1}}] \\ + \frac{dv}{dx}[\beta_{K-2}v^* + \beta_{K-3}\frac{dv^*}{dx} + ... + (-1)^{2K-2}\beta_0 \frac{d^{K-2}v^*}{dx^{K-2}}] \\ + ... + (-1)^K \frac{d^{K-1}v}{dx^{K-1}}\beta_0 v^* \end{aligned} \tag{19}$$

Before we proceed further, it may be noted that if the operator L_1 consists of only odd derivatives, then

$$L_1 + L_1^* = 0 \tag{20}$$

and

$$P_1(v , v^*) + P_2(v , v^*) = 0 \tag{21}$$

In this special case a proper eigenvalue problem would be defined by $Lv = \omega v$ with the boundry conditions $L_j v = 0$, $j = 1 , 2 , ... , N$, which, incidentally, coincides with the equation of motion of free vibration and is obviously self-adjoint. It may be further observed that in a general situation $(L_1 + L_1^*)$ together will cancel all odd derivatives and will consist of terms with only even derivatives. Similarly, $(P_1 + P_2)$ will also cancel terms with coefficients β_{K-1} , β_{K-3} , ... and, therefore, in the following proof we will insist that L_1 consists of only even derivatives and K is an even integer. Thus, a close examination of (18-19) reveals that

$$P_1 = P_2 \tag{22}$$

and (17) reduces to

$$\langle v^* , (l + \beta L_1 + \beta L_1^*)v\rangle = \langle v , (L + \beta L_1^* + \beta L_1)v^*\rangle$$
$$+ [P(v , v^*)]_a^b + 2\beta[P (v , v^*)]_a^b \tag{23}$$

The bracketed terms on the right side of (23) may now be expressed as

$$[P(v\ ,\ v^*)]_a^b + 2\beta[P_1(v\ ,\ v^*)]_a^b = \sum_{n=1}^{2N} (L_n + \beta M_n)v(L_{2N+1-n} + \beta M_{2N+1-n})v^* \tag{24}$$

and (23) yields

$$\langle v^*\ ,\ (\mathcal{L} + \beta\mathcal{L}_1 + \beta\mathcal{L}_1^*)v\rangle - \langle v\ ,\ (\mathcal{L} + \beta\mathcal{L}_1 + \beta\mathcal{L}_1^*)v^*\rangle$$

$$= \sum_{n=1}^{2N} (L_n + \beta M_n)v(L_{2N+1-n} + \beta M_{2N+1-n})v^* \tag{25}$$

Thus if (16) is obeyed by v and v^*, the right hand side of (25) is zero. Therefore, the theorem is proved.

When operator $\mathcal{L}_1$ consists of both odd and even derivatives, a proper selection of a self-adjoint system would consist of the same equations (15) and (16) with the restrictions that terms involving β_{K-1}, β_{K-3}, ... are identically zero. This follows in a natural way from the above proof.

GENERAL CASE WITH $\mathcal{L}_1$ HAVING VARIABLE COEFFICIENTS

In the previous section it was possible to study the properties of the bilinear concomitants P_1 and P_2 in explicit forms which facilitated the proof. For a more general operator $\mathcal{L}_1$ having variable coefficients, it will be seen now that an indirect approach must be adopted. A wide group of problems in structural stability with distributed follower loadings gives rise to the study of eigenvalue problems in which the operator $\mathcal{L}_1$ involves variable coefficients. It has been experienced that distributed follower loading is not the only source of the origin of variable coefficients, but several other problems such as the instability in bending-torsion of a rectangular bar acted upon by a transverse follower force, also have a similar nature. Thus, it seems desirable to extend the results of the previous section to encompass

a wider group. Accordingly, we consider the following form of L_1 along with (1) and (4)

$$L_1 = \sum_{n=1}^{K} \beta_{K-n}(x) \frac{d^n}{dx^n} \quad , \quad K \leq N \tag{26}$$

The $\beta_n(x)$'s are continuous functions of x whose $K-n$ derivatives with respect to x exist and are continuous.

We note the following relation:

$$\langle u^* , L_1 u\rangle = \langle u , L_1^* u^*\rangle + [P_3(u , u^*)]_a^b \tag{27}$$

where

$$L_1^* u^* = (-1)^K \frac{d^K(\beta_0 u^*)}{dx^K} + (-1)^{K-1} \frac{d^{K-1}(\beta_1 u^*)}{dx^K} + \ldots\ldots\ldots - \frac{d(\beta_{K-1} u^*)}{dx} \tag{28}$$

and P_3 is a bilinear form in $u(a)$, $u'(a)$, ... , $u^{(K-1)}(a)$, $u(b)$, ... , $u^{(K-1)}(b)$, $u^*(a)$, ... , $u^*(a)$, ... , $u^{*(K-1)}(b)$. Analogous to the previous case, P_3 may be expressed as

$$[P_3(u , u^*)]_a^b = \sum_{n=1}^{2K} L_n u M_{2K+1-n} u^* \tag{29}$$

where M_1 , M_2 , ... M_{2K} are linearly independent forms in $u^{*i}(a)$ and $u^{*i}(b)$, $i = 0 , 1 , \ldots\ldots K$. On the other hand, we have

$$\langle u^* , L_1^* u\rangle = \langle u , L_1 u^*\rangle + [P_4(u , u^*)]_a^b \tag{30}$$

The bilinear concomitants P_3 and P_4 have the following forms in u^i and u^{*i}

$$P_3(u\ ,\ u^*) = u[\beta_{K-1}u^* - \frac{d(\beta_{K-2}u^*)}{dx} + \ldots\ldots. + (-1)^{K-1}\frac{d^{K-1}(\beta_0 u^*)}{dx^{K-1}}]$$
$$+ \frac{du}{dx}[\beta_{K-2}u^* - \frac{d(\beta_{K-3}u^*)}{dx} + \ldots.. + (-1)^{K-2}\frac{d^{K-2}(\beta_0 u^*)}{dx^{K-2}}]$$
$$+ \ldots\ldots\ldots\ldots\ldots + \frac{d^{K-1}u}{dx^{K-1}}\beta_0 u^* \tag{31}$$

and

$$P_4(u\ ,\ u^*) = u^*[-\beta_{K-1}u + \frac{d(\beta_{K-2}u)}{dx} + \ldots.. + (-1)^K\frac{d^{K-1}(\beta_0 u)}{dx^{K-1}}]$$
$$+ \frac{du^*}{dx}[-\beta_{K-2}u + \frac{d(\beta_{K-3}u)}{dx} + \ldots\ldots + (-1)^{K-1}\frac{d^{K-2}(\beta_0 u^*)}{dx^{K-2}}]$$
$$+ \ldots\ldots + (-1)^{2K-1}\frac{d^{K-1}u^*}{dx^{K-1}}\beta_0 u \tag{32}$$

Thus we see that (32) may be obtained from (31) after replacing u by u^* and u^* by $-u$, and vice versa. Consequently, a counter part of (29) may be written for P_4 by

$$[P_4(u\ ,\ u^*)]_a^b = \sum_{n=1}^{2K} M_n u L_{2K+1-n} u^* \tag{33}$$

We now show that the eigenvalue problem given by the differential equation

$$Lv + \beta L_1 v + \beta L_1^* v = \omega v \tag{34}$$

with the boundary conditions

$$L_j v + \beta M_j v = 0 \quad , \quad j = 1\ , 2\ , \ldots..\ , N \tag{35}$$

is self-adjoint.

As we have discussed, one obtains

$$<v^* \ , \ (L + \beta L_1 + \beta L_1^*)v> = <v \ , \ (L + \beta L_1^* + \beta L_1)v^*> + [P(v \ , \ v^*)]_a^b$$
$$+ \beta[P_3(v \ , \ v^*)]_a^b + \beta[P_4(v \ , \ v^*)]_a^b \tag{36}$$

which yields, after using (29) and (33)

$$<v^* \ , \ (L + \beta L_1 + \beta L_1^*)v> - <v \ , \ (L + \beta L_1 + \beta L_1^*)v^*>$$
$$= \sum_{n=1}^{2N} (L_n + \beta M_n)v(L_{2N+1-n} + \beta M_{2N+1-n})v^* \tag{37}$$

and if (35) is obeyed, the right hand side of (37) vanishes. This concludes the proof.

RELATIONSHIPS

In view of the fact that the system governed by (34-35) is self-adjoint, it is capable of yielding accurate solutions based upon the usual Galerkin procedure or other approximation methods. Therefore, it is of interest to investigate if this system has any relationship with the original problem. Indeed, it will be seen in the next section that, based upon certain relationships discussed in the following, an extension of the Galerkin method for the solution of the original nonconservative system may be advanced which has various desirable features.

Let us consider the following boundary value problems:

$$Lu + \beta L_1 u = f \tag{38a}$$

$$L_j u = 0 \quad , \quad j = 1 \ , 2 \ , \ldots\ldots \ , N \tag{38b}$$

and

$$\mathcal{L}v + \beta \mathcal{L}_1 v + \beta \mathcal{L}_1^* v = f \tag{39a}$$

$$L_j v + \beta M_j v = 0 \quad , \quad j = 1 , 2 , \ldots\ldots , N \tag{39b}$$

where f is an arbitrary, continuous function of the variable x. Multiply (38a) by v and integrate in the domain of x to obtain

$$\langle v , (\mathcal{L} + \beta \mathcal{L}_1)u \rangle = \langle u , (\mathcal{L} + \beta \mathcal{L}_1^*)v \rangle = \langle v , f \rangle \tag{40}$$

which yields after using (39)

$$\langle u , \beta \mathcal{L}_1 v \rangle = \langle u , f \rangle - \langle v , f \rangle \tag{41}$$

Similarly, starting with (39a) and using (38) we obtain

$$\langle v , \beta \mathcal{L}_1^* u \rangle = \langle u , f \rangle - \langle v , f \rangle \tag{42}$$

Thus by comparing (41) and (42) we have

$$\langle u , \mathcal{L}_1 v \rangle = \langle v , \mathcal{L}_1^* u \rangle \tag{43}$$

which is reminiscent of the Betti-Rayleigh reciprocity relation in classical solid mechanics. Note that the operator $\mathcal{L}_1$ gives the field of nonconservative loading and a spatial average of the type $\langle u , \mathcal{L}_1 v \rangle$ represents work done by $\mathcal{L}_1 v$ through displacement u.

It may be pointed out here that a more direct relationship exists with the adjoint boundary value problem. Along with (38) consider the following

$$\mathcal{L}u^* + \beta \mathcal{L}_1^* u^* = f \tag{44a}$$

$$L_j u^* + \beta M_j u^* = 0 \quad , \quad j = 1 , 2 , \ldots\ldots , N \tag{44b}$$

Multiplying (38a) by u^* and integrating in the domain of x, we obtain

$$\langle u^* , (\mathcal{L} + \beta \mathcal{L}_1) u \rangle = \langle u , (\mathcal{L} + \beta \mathcal{L}_1^*) u^* \rangle = \langle u^* , f \rangle \tag{45}$$

If we now use (44a), (45) yields

$$\langle u , f \rangle = \langle u^* , f \rangle \tag{46}$$

which states: The spatial averages of the products of the external loading with the displacements of a nonconservative system and its adjoints are the same.

AN EXTENSION OF THE GALERKIN METHOD

In the theory of stability wide use is made of approximate methods to determine the regions of stability of a particular problem. Accurate determination of critical parameters which separate stable from unstable regions of a conservative system may be facilitated by the use of a particular technique available, (e.g. Methods of Ritz, Rayleigh, Galerkin, etc.) all of which have their origins in variational principles [1].

Several extensions of these methods have been suggested for nonconservative systems which, however, are only heuristically justifiable and lack proofs of convergence and estimations of error involved. Most authors analyze

nonconservative stability problems in two parts. First, a related simpler eigenvalue problem, (usually governing equations of free vibration which constitutes a self-adjoint system, see Bolotin [2]),which yields a complete set of trial functions, is solved. Then, with the help of this set of trial functions, an application of the approximate procedure adopted reduces the problem to determining the roots of a certain characteristic determinant. However, since no firm guide line in the selection of the trial functions existed, the establishment of convergence and an estimation of the error involved is not readily possible.

In the following we suggest an extension of the Galerkin method of solving the general eigenvalue problem

$$\mathcal{L}u + \beta\mathcal{L}_1 u = \lambda u \tag{46a}$$

$$L_j u = 0 \quad , \quad j = 1 , 2 , \ldots\ldots , N \tag{46b}$$

using the eigenfunctions of the self-adjoint system defined by

$$\mathcal{L}v_j + \beta\mathcal{L}_1 v_j + \beta\mathcal{L}_1^{*} v_j = \omega_j v_j \tag{47a}$$

$$L_m v_j + \beta M_m v_j = 0 \quad , \quad m = 1 , 2 , \ldots\ldots , N \tag{47b}$$

We denote by $\{\omega_i\}$ and $\{v_i\}$ the sets of eigenvalues and eigenfunctions, respectively, of (47) and consider the following integral

$$\langle v_j \, , \, (\mathcal{L}u + \beta\mathcal{L}_1 u)\rangle = \lambda\langle v_j \, , \, u\rangle$$

which, after integration by parts and in view of (47), reduces to

$$\langle u\ ,\ \beta L_1^{*} v_j\rangle = (\omega_j - \lambda)\langle u\ ,\ v_j\rangle \tag{48}$$

As is evident from (48), we have converted the differential equation to the simplest kind of an integral relation. In fact, this integral equation necessitates determining the area under a curve given by the unknown function u which is multiplied by certain known functions involving v_j and their derivatives. Thus, in accordance with practice in numerical analysis, the integration may be replaced by the summation of a finite series in terms of the ordinates of the function u at certain intervals. There are several well known rules of carrying out this type of numerical integration for which an estimate of the round-off error may be obtained. In order to illustrate the procedure of the proposed method, we obtain in the following a general quadrature relation based on Newton's interpolation formula. Let u take the values u_0 , u_1 , ... , u_N for the equidistant points x_0 , x_1 , ... , x_N of the independent variable x with $x_0 = a$ and $x_N = b$, the interval being h . In Newton's formula we have

$$x = x_0 + hn$$

from which we get

$$dx = hdn$$

and in terms of forward differences

$$u(x) = u(x_0 + hn) = u_0 + n\Delta u_0 + \frac{n(n-1)}{2!}\Delta^2 u_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 u_0 + \ldots\ldots + \frac{n(n-1)\ldots(n-N+1)}{N!}\Delta^N u_0 \tag{49}$$

where

$$\Delta u_0 = u_1 - u_0$$

$$\Delta^2 u_0 = \Delta u_1 - \Delta u_0 = u_2 - 2u_1 + u_0$$

$$\Delta^3 u_0 = \Delta^2 u_1 - \Delta^2 u_0 = u_3 - 3u_2 + 3u_1 - u_0$$

etc.

Substituting (49) in (48) and carrying out the integration with respect to n, the following expression is obtained

$$[Na_{0j}u_0 + \frac{N^2}{2} a_{1j}\Delta u_0 + (\frac{N^3}{3} - \frac{N^2}{2})a_{2j}\frac{\Delta^2 u_0}{2!} + (\frac{N^4}{4} - N^3 + N^2)a_{3j}\frac{\Delta^3 u_0}{3!}$$

$$+ (\frac{N^5}{5} - \frac{3N^4}{2} + \frac{11N^3}{3} - 3N^2)a_{4j}\frac{\Delta^4 u_0}{4!} + \dots\dots\dots]$$

$$= (\omega_j - \lambda)[Nb_{0j}u_0 + \frac{N^2}{2} b_{1j}\Delta u_0 + (\frac{N^3}{3} - \frac{N^2}{2})b_{2j}\frac{\Delta^2 u_0}{2!} + (\frac{N^4}{4} - N^3 + N^2)b_{3j}\frac{\Delta^3 u_0}{3!}$$

$$+ (\frac{N^5}{5} - \frac{3N^4}{2} + \frac{11N^3}{3} - 3N^2)b_{4j}\frac{\Delta^4 u_0}{4} + \dots\dots] \qquad (50)$$

where

$$a_{nj} = \beta L_1^* v_j(x_0 + hn)$$

and

$$b_{nj} = v_j(x_0 + hn)$$

Keeping in mind that $\Delta^k u_0$ may be expressed in terms of u_0, u_1, ..., u_k equation (50) may be arranged in the following form

$$E_N = \frac{h^{N+1} u^{(N+1)}(3)}{(n+1)!} n(n-1)\ldots(n-N) \quad , \quad a < 3 < b \tag{53}$$

Thus, we see that the round-off error E_N may be made as small as we please by making the interval smaller. Estimates of the error involved in the methods of Simpson, Weddle, Gauss, etc. are discussed in [7] and [8].

The above scheme of numerical treatment does not require integration of any kind, by contrast to the commonly employed approximate methods and, therefore, is suitable for direct use on a digital computer. It may be mentioned that the same scheme may be found more convenient also for self-adjoint systems, in which case v in (48) may be any complete set of functions satisfying the same boundary conditions as those for u. In a forthcoming study, numerical results of several nonconservative stability problems will be presented.

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